MATHEMATICS OF TAX-ADVANTAGED RETIREMENT SAVINGS – PART 1

1. How do you imagine most people pay expenses during their retirement, after they have stopped working and no longer receive a wage?

Almost everyone needs to save for retirement. There are a number of ways to do this. How can we make sense of the options we have and choose the best one? First, we need to provide some background and then we will get into the mathematics that can help us make these important decisions.

Background on Saving for Retirement

Saving for retirement is an important financial goal for most people.

- Social Security benefits are not likely to be enough to cover the expenses most people will face in retirement.
- Pensions that pay an annual amount for one's entire retirement used to be common but are generally not provided by employers anymore.

To encourage retirement savings the government offers tax-based incentives for people to contribute to their retirement accounts.

Tax is an especially important consideration when saving for retirement. You potentially face two kinds of tax: first, a tax on your wages when they are paid and, second, a tax on the interest your investments earn.

Example: You invest \$5000 for 20 years, earning an annual interest of 7%. How much would you have if there were no taxes?

- $FV = 5000(1.07)^{20} = 19{,}300$

Unfortunately, there is almost always tax! What how much would you have if a tax of 20% is applied to both the initial investment (your wages) and to the interest you earn each year?

- Your initial investment will now be reduced in value by 20%, $5000 \bullet 0.8 = \$4000$
- After one year, your investment will have value $4000 \bullet 1.07 = \$4280$ before taxes
- You now pay tax on the *interest income* – that's the amount it increased by that year, not the entire amount, in this example your interest income in the first year will be $4280 - 4000 = \$280$. So you are taxed 20% of \$280 which is \$56 in taxes.
- After one year the final value of your investment will $4280 - 56 = \$4224$.

- If we work backwards, we can break down all our work, and rearrange the mathematics to see what's going on:

$$\begin{aligned} & 4224 \\ & = 4280 - 56 \\ & = 4280 - (280 \bullet 0.2) \\ & = 4000 \bullet 1.07 - 4000 \bullet 0.07 \bullet 0.2 \\ & = 4000 \bullet (1.07 - 0.07 \bullet 0.2) \\ & = 4000 \bullet (1 + 0.07(1 - 0.2)) \\ & = (5000 \bullet 0.80) \bullet (1 + 0.07(1 - 0.2)) \end{aligned}$$

- Notice that whatever the initial amount c invested, after one year, the value after tax will be $c \bullet (1 + 0.07(1 - 0.2))$, in other words the tax makes you effectively earn interest at a lower rate, $0.07(1 - 0.2) = 0.056$. So even though you are earning 7%, you are only really earning 5.6% after taking taxes into account.
- This means that after 20 years, the value of your investment will be worth $FV = 4000 \bullet 1.056^{20} = \$11{,}900$

Tax brings down the future value of your investment from \$19,300 to \$11,900 – a significant drop-off! In order to encourage retirement savings, the government provides some tax breaks which allow individuals to avoid paying some of these taxes provided they are saving that money to use in their retirement.

Tax-Incentive Based Retirement Accounts

There are two types of tax-incentive based retirement accounts we will discuss initially:

1. Individual Retirement Account (IRA, often known as a "traditional IRA")
   - You pay *no* tax on your salary when it goes into the account.
   - You pay *no* tax on income from interest earned in this account.
   - You will pay tax when you withdraw from this account during retirement.
2. The Roth IRA
   - You will pay tax on your salary, even when it goes into the account.
   - You pay *no* tax on income from interest earned in this account.
   - You will pay *no* tax when you withdraw from this account during retirement.

Key facts to note:

- In all the tax-incentive based retirement accounts you pay no tax on income from interest.
- In all tax-incentive based retirement accounts withdrawals prior to retirement are taxed as income and also face an additional penalty for early withdrawal.
- In a traditional IRA, you pay no tax on your salary when it goes into the account but you pay tax when you withdraw it at retirement.

- In a Roth IRA, you pay tax on your salary when it goes into the account but you pay no tax when you withdraw it at retirement.

See https://www.irs.gov/retirement-plans for more details and other types of retirement plans, you can also use this QR code to access it →

While we use the terms pre-tax and post-tax, others use the terms before-tax and after-tax to refer to the same concepts.

How Big of an Advangage Do Tax-Advantaged Retirement Accounts Provide?

We saw above that if there is a tax rate of 20% and return of 7%, the future value of a $5000 taxable investment is:

$$FV = (5000 \bullet 0.8) \bullet (1 + 0.07 \bullet 0.8)^{20} = \$11{,}900$$

With a traditional IRA, there is no tax on the investment income and no tax at the time of your contribution, so prior to withdrawal, the value is $5000(1.07)^{20}$ =19,300, but you are taxed at 20% when you withdraw the money at retirement, so the amount you have to spend in retirement after taxes is:

$$FV = 5000(1.07)^{20} \bullet 0.8 = \$15{,}500$$

How much more do you have with a traditional IRA than a taxable investment? We can find how much more, as a percent, by taking the ratio of the two values:

$$\frac{15{,}500}{11{,}900} = 1.3$$

In other words, in the above example, your investment is worth approximately 30% more with a traditional IRA than a taxable investment.

We can also arrive at this number using the original algebraic expressions and simplifying:

$$\frac{5000(1.07)^{20} \bullet 0.8}{(5000 * 0.8) * (1 + 0.07 \bullet 0.8)^{20}}$$

$$\frac{\cancel{5000}(1.07)^{20} \bullet \cancel{0.8}}{(\cancel{5000} \bullet \cancel{0.8}) \bullet (1 + 0.07 \bullet 0.8)^{20}}$$

$$\left(\frac{1.07}{1 + 0.07 \bullet 0.8}\right)^{20} = 1.3$$

Again, arriving at the same ratio, showing that in this example your investment would be worth approximately 30% more with a traditional IRA than a taxable investment.

However, this is just one example. Using a similar approach we can explore these same ideas to generalize.

Making Sense of Tax-Advantaged Retirement Accounts

To start, we will assume that a person will have the same tax rate over their entire life. We will use the Greek letter Tau, τ, to represent the tax rate. Later we will look at the effect of different tax rates.

We assume that investments provide an annual return of r and that the investment horizon until retirement (the amount of time the money will be invested prior to retirement) is T years.

Now let's look at what happens to the value of $1 of earnings that is invested.

To model investing the dollar directly, without using an IRA account we can use the following equation:

$$\text{Taxable Investment: } (1-\tau)(1+r(1-\tau))^T$$

- τ represents the tax rate
- r represents the annual return
- T represents the number of years till retirement
- The first $(1-\tau)$ represents amount of money left to invest after income taxes.
- The second $(1-\tau)$ represents the percentage of investment return that is left after taxes.

To model investing our dollar in an IRA we can use the following equation:

$$\text{Individual Retirement Account (IRA): } (1+r)^T(1-\tau)$$

- τ represents the tax rate
- r represents the annual return
- T represents the number of years till retirement
- $(1-\tau)$ represents the percentage of investment return that is left after taxes.

With an IRA the only time you pay tax is when withdrawing money.

We can look at the ratio of these two expressions to see the tax advantage of the IRA account. What is the IRA's advantage? The ratio of the value of an IRA to the value of a taxable account can help us. It is on the left. It mirrors the numerical example from above and the two examples have been juxtaposed to help you follow along. The numerical example is on the right. The principal investment in the numerical example from above has been changed from $5000 to $1 so as to better mirror our general argument:

General Case	Numerical Example
$\frac{(1+r)^T(1-\tau)}{(1-\tau)(1+r(1-\tau))^T}$	$\frac{(1.07)^{20} \bullet 0.8}{0.8 \bullet (1+0.07 \bullet 0.8)^{20}}$

We can simplify this expression. The terms $(1-\tau)$ can be eliminated from the numerator and denominator:

General Case	Numerical Example
$\frac{(1+r)^T\cancel{(1-\tau)}}{\cancel{(1-\tau)}(1+r(1-\tau))^T} = \frac{(1+r)^T}{(1+r(1-\tau))^T}$	$\frac{(1.07)^{20} \bullet \cancel{0.8}}{\cancel{0.8} \bullet (1+0.07 \bullet 0.8)^{20}} = \frac{1.07^{20}}{(1+0.07 \bullet 0.8)^{20}}$

The numerator and denominator are raised to the same power so they can be combined as well.

General Case	Numerical Example
$\left[\frac{(1+r)}{(1+r(1-\tau))}\right]^T$	$\left(\frac{1.07}{1+0.07 \bullet 0.8}\right)^{20} = 1.3$

Since $\tau < 1$, we know that $r(1-\tau) < r$. From this we can deduce that the advantage the IRA offers is that *returns on investments are not taxed* (as noted in the bullets above in the section introducing retirement accounts). The advantage does not come from the fact that the contributions are not taxed.

This is somewhat counter intuitive. Without doing the math it would seem that if you have more money to invest, you will make more for retirement. However, since the money will eventually be taxed when withdrawn all that matters is the rate of growth of the money while it is invested.

	Taxable Investment	Traditional IRA	Roth IRA
Are contributed earnings taxed?	YES	NO	YES
Are investment earnings taxed?	YES	NO	NO
Are withdrawals taxed after retirement?*	NO	YES	NO

*There are fees if withdrawals from IRAs are made prior to retirement.

Comparing Outcomes

The above equations track the value of a single dollar. To track the value of a greater principal investment of P dollars we need to include this in our equations.

$$\text{Taxable Investment: } P(1-\tau)(1+r(1-\tau))^T$$

$$\text{Traditional IRA: } P(1+r)^T(1-\tau)$$

$$\text{Roth IRA : } P(1-\tau)(1+r)^T$$

- P represents the principal investment
- τ represents the tax rate

- r represents the annual return
- T represents the number of years till retirement

Example: Suppose you invest \$7,000 at 8% annual interest for 36 years and you have a 25% tax rate. How much money would you have with a taxable investment vs. IRA investment when you withdraw the money during retirement?

Taxable Investment:	Individual Retirement Account (IRA):
$P(1-\tau)\big(1+r(1-\tau)\big)^T$ $= \$7{,}000(1-0.25)\big(1+0.08(1-0.25)\big)^{36}$ $= \$42{,}773.07$	$P(1-\tau)(1+r)^T$ $= \$7{,}000(1-0.25)(1+0.08)^{36}$ $= \$83{,}832.90$

Just plugging in the numbers hides the math. The difference between the two calculations is the investment return *after* we account for taxes. If you calculate the investment return after taking taxes into account, the IRA the investment return is 8% and in the taxable investment the investment return is only 6%. This is because earnings on investments in IRAs and Roth IRAs are not taxed as they would be in a normal investment.

Taxable Investment:	Individual Retirement Account (IRA):
$P(1-\tau)\big(1+r(1-\tau)\big)^T$ $= \$7{,}000(1-0.25)\big(1+\mathbf{0.08}(\mathbf{1}-\mathbf{0.25})\big)^{36}$ $= \$42{,}773.07$ $0.08\,(1-0.25) = 0.08(0.75) = 0.06 = \mathbf{6}\%$	$P(1-\tau)(1+r)^T$ $= \$7{,}000(1-0.25)(1+\mathbf{0.08})^{36}$ $= \$583{,}832.90$ $0.08 = \mathbf{8}\%$

Using the rule of 72 we can see that money in the IRA will double in approximately 9 years, but in the taxable account will take 12 years to double.

Taxable Investment:	Individual Retirement Account (IRA):
$\frac{72}{6} = 12$ An investment with a return of 6% will take approximately *12 years* to double in value.	$\frac{72}{8} = 9$ An investment with a return of 8% will take approximately *9 years* to double in value.

Over 36 years, the amount in the taxable account can double 3 times, while the amount in the IRA can double 4 times.

2. What was the post tax investment for \$7000? What multiple of this amount were the proceeds from the Taxable Investment and the IRA? How is this related to the rule of 72?

For questions 3-5, suppose you invest $5,000 at 7.2% annual interest for 30 years and you have a 33% tax rate. You can choose to invest in a taxable investment account or an IRA.

3. How much money would you have with a taxable investment vs. IRA when you withdraw the money during retirement?

Taxable Investment:	Individual Retirement Account (IRA):

4. What is your investment return after taking taxes into account with a taxable investment? What is your investment return after taking taxes into account on an IRA?

5. Using the values, you calculated in the previous question and the rule of 72, approximately how much would you expect to have if you put your money in a taxable investment? Approximately how much would you have if you put it in an IRA?

Roth IRA vs Traditional IRA

The only difference between the IRA and traditional IRA is the order of the terms since a Roth IRA taxes money on the way in and a Traditional IRA taxes money on the way out.

$$\text{Traditional IRA: } (1+r)^T(1-\tau)$$

$$\text{Roth IRA : } (1-\tau)(1+r)^T$$

Mathematically we can see if the tax rate is the same during working years and retirement than there is no difference in outcome between the two types of IRAs!

6. What is the mathematical property that allows us to see that $(1+r)^T(1-\tau) = (1-\tau)(1+r)^T$? (Or that A•B = B•A?)

From the equations above we know that either form of IRA will have the same answer *as long as the tax rate is the same* while working and when retiring.

7. Do you suspect your tax rate will remain the same or change over the course of your financial life cycle? Why might one's tax rate change at they move into their retirement?

8. If your tax rates change between working and retirement do you think this will change things? Why or why not?

Comparing IRAs and Company 401(k)s

Another third common type of retirement account is a **Company 401(k)**. Unlike an IRA, which an individual can set up for themselves, this account must be set up by an employer. Here are some further details.

- These plans must be offered by your employer.
- Companies may offer direct contributions to 401(k) accounts or match your contributions.
- These plans will generally act like IRAs, in that deposits are not taxed, but some act like Roth IRAs, in which deposits are taxed but withdrawals are not.
- Company 401(k)s allow a much higher maximum contribution per year than IRAs.
- Like with all IRAs, you will pay a penalty if you withdraw money from a 401(k) early.

The math of tax benefits for 401(k)s is exactly the same as with IRAs, the difference is who sets them up, and also that the company may make additional contributions to the account.

Example: Banana Corp. offers a company 401(k) with which it matches employee contributions up to a maximum of $5000 per year (before tax).

- If you contribute $1000, the company will contribute $1000
- If you contribute $5000, the company will contribute $5000
- If you contribute $7000 (or any amount greater than $5000), the company will still only contribute $5000

9. If you contribute $3500 to Banana corp.'s 401(k), with a return of 8.5% and a tax rate of 19%, what will the post tax value of your investment be after 25 years?

10. Does the fact a company is matching your 401(k) contributions motivate you to contribute more or less to the account? Explain why.

We will look at the mathematics behind making these decisions when your tax rate changes from work till retirement next.